

# Kleene Algebras and Logic: Boolean and Rough Set Representations, 3-valued, Rough and Perp Semantics

Arun Kumar and Mohua Banerjee

Department of Mathematics and Statistics,  
Indian Institute of Technology, Kanpur 208016, India  
arunk2956@gmail.com, mohua@iitk.ac.in

**Abstract.** A structural theorem for Kleene algebras is proved, showing that an element of a Kleene algebra can be looked upon as an ordered pair of sets. Further, we show that negation with the Kleene property (called the ‘Kleene negation’) always arises from the set theoretic complement. The corresponding propositional logic is then studied through a 3-valued and rough set semantics. It is also established that Kleene negation can be considered as a modal operator, and enables giving a perp semantics to the logic. One concludes with the observation that all the semantics for this logic are equivalent.

**Key words:** Kleene algebras, Rough sets, 3-valued logic, Perp semantics.

## 1 Introduction

Algebraists, since the beginning of work on lattice theory, have been keenly interested in representing lattice-based algebras as algebras based on *set* lattices. Some such well-known representations are the Birkhoff representation for finite lattices, Stone representation for Boolean algebras, or Priestley representation for distributive lattices. It is also well-known that such representation theorems for classes of lattice-based algebras play a key role in studying set-based semantics of logics ‘corresponding’ to the classes. In this paper, we pursue this line of investigation, and focus on *Kleene algebras* and their representations. We then move to the corresponding propositional logic, denoted  $\mathcal{L}_K$ , and define a 3-valued, rough set and perp semantics for it. Through the work here, one is able to establish that  $\mathcal{L}_K$  and the 3-element Kleene algebra **3** (cf. Figure 1, Section 2) play the same fundamental role among the Kleene algebras that classical propositional logic and the 2-element Boolean algebra **2** play among the Boolean algebras.

Kleene algebras were introduced by Kalman [1] and have been studied under different names such as normal i-lattices, Kleene lattices and normal quasi-Boolean algebras e.g. cf. [2, 3]. The algebras are defined as follows.

**Definition 1.** *An algebra  $\mathcal{K} = (K, \vee, \wedge, \sim, 0, 1)$  is called a Kleene algebra if the following hold.*

1.  $\mathcal{K} = (K, \vee, \wedge, \sim, 0, 1)$  is a De Morgan algebra, i.e.,
  - (i)  $(K, \vee, \wedge, 0, 1)$  is a bounded distributive lattice, and for all  $a, b \in K$ ,
  - (ii)  $\sim(a \wedge b) = \sim a \vee \sim b$  (De Morgan property),
  - (iii)  $\sim\sim a = a$  (involution).
2.  $a \wedge \sim a \leq b \vee \sim b$ , for all  $a, b \in K$  (Kleene property).

In order to investigate a representation result for Kleene algebras, it would be natural to first turn to the known representation results for De Morgan algebras, as Kleene algebras are based on them. One finds the following, in terms of sets.

- Rasiowa [4] represented De Morgan algebras as set-based De Morgan algebras, where De Morgan negation is defined by an involution function.
- In Dunn's [5, 6] representation, each element of a De Morgan algebra can be identified with an ordered pair of sets, where De Morgan negation is defined as reversing the order in the pair. We note that this representation of De Morgan algebras leads to Dunn's famous 4-valued semantics of De Morgan logic.

On the other hand, we also find that there are algebras *based on* Kleene algebras which can be represented by ordered pairs of sets, and where negations are described by set theoretic complements. Consider the set  $B^{[2]} := \{(a, b) : a \leq b, a, b \in B\}$ , for any partially ordered set  $(B, \leq)$ .

- (Moisil (cf. [7])) For each 3-valued Łukasiewicz algebra  $\mathcal{A}$ , there exists a Boolean algebra  $B$  such that  $\mathcal{A}$  can be embedded into  $B^{[2]}$ .
- (Katriňák [8], cf. [9]) Every regular double Stone algebra can be embedded into  $B^{[2]}$  for some Boolean algebra  $B$ .

Rough set theory [10, 11] also provides a way to represent algebras as pairs of sets. In rough set terminology (that will be elaborated on in Section 4), we have the following.

- (Comer [12]) Every regular double Stone algebra is isomorphic to an algebra of rough sets in a Pawlak approximation space.
- (Järvinen [13]) Every Nelson algebra defined over an algebraic lattice is isomorphic to an algebra of rough sets in an approximation space based on a quasi order.

In this article, the following representation results are established for Kleene algebras.

**Theorem 1.**

- (i) Given a Kleene algebra  $\mathcal{K}$ , there exists a Boolean algebra  $\mathcal{B}_{\mathcal{K}}$  such that  $\mathcal{K}$  can be embedded into  $\mathcal{B}_{\mathcal{K}}^{[2]}$ .
- (ii) Equivalently, every Kleene algebra is isomorphic to an algebra of rough sets in a Pawlak approximation space.

The De Morgan negation operator with the Kleene property mentioned in Definition 1, is referred to as the *Kleene negation*. In literature, one finds various generalizations of the classical (Boolean) negation, including the De Morgan and Kleene negations. It is natural to ask the following question: do these generalized negations arise from (or can be described by) the Boolean negation? The representation result above (Theorem 1) for Kleene algebras shows that Kleene algebras always arise from Boolean algebras, thus answering the above question in the affirmative for the Kleene negation.

We next proceed to study the logic  $\mathcal{L}_K$  corresponding to the class of Kleene algebras.  $\mathcal{L}_K$  is the De Morgan consequence system [6] with the negation operator satisfying the *Kleene axiom*:  $\alpha \wedge \sim \alpha \vdash \beta \vee \sim \beta$ . We show that  $\mathcal{L}_K$  is sound and complete with respect to a 3-valued as well as a rough set semantics. Moreover, it can be imparted a *perp semantics* [14]. In fact, the perp semantics provides a framework to study various negations from the logic as well as algebraic points of view. In particular, De Morgan logic is sound and complete with respect to a class of perp (*compatibility*) frames, and thus the algebraic semantics, 4-valued semantics and perp semantics for De Morgan logic coincide. In case of the logic  $\mathcal{L}_K$ , we obtain that its algebraic, 3-valued, rough set and perp semantics are all equivalent.

The paper is organized as follows. In Section 2, we prove (i) of Theorem 1, in the form of Theorem 2. The logic  $\mathcal{L}_K$  and its 3-valued semantics are introduced in Section 3, further we prove soundness and completeness results. In Section 4, we establish a rough set representation of Kleene algebras, that is, (ii) of Theorem 1, relate rough sets with the 3-valued semantics considered in this work, and finally present completeness of  $\mathcal{L}_K$  with respect to the rough set semantics. In Section 5, we discuss the perp semantics for  $\mathcal{L}_K$ , and investigate the Kleene property in perp frames. Section 5 ends with the observation that all the semantics defined for  $\mathcal{L}_K$  are equivalent (Theorem 20). We conclude the article in Section 6.

The lattice theoretic results used in this article are taken from [15]. We use the convention of representing a set  $\{x, y, z, \dots\}$  by  $xyz\dots$ .

## 2 Boolean representation of Kleene algebras

Construction of new types of algebras from a given algebra has been of prime interest for algebraists, especially in the context of algebraic logic. Some well known examples of such construction are:

- Nelson algebra from a given Heyting algebra (Vakarelov [16, 17]).
- Kleene algebras from distributive lattices (Kalman [1]).
- 3-valued Łukasiewicz-Moisil (LM) algebra from a given Boolean algebra (Moisil, cf. [7]).
- Regular double Stone algebra from a Boolean algebra (Katriňák [8], cf. [9]).

Our work is based on the Moisil construction of a 3-valued LM algebra (which is, in particular, a Kleene algebra). Let us present this construction.

Let  $\mathcal{B} := (B, \vee, \wedge, ^c, 0, 1)$  be a Boolean algebra. Consider again, the set

$$B^{[2]} := \{(a, b) : a \leq b, a, b \in B\}.$$

**Proposition 1.**  $\mathcal{B}^{[2]} := (B^{[2]}, \vee, \wedge, \sim, (0, 0), (1, 1))$  is a Kleene algebra, where,

$$(a, b) \vee (c, d) := (a \vee c, b \vee d),$$

$$(a, b) \wedge (c, d) := (a \wedge c, b \wedge d),$$

$$\sim(a, b) := (b^c, a^c).$$

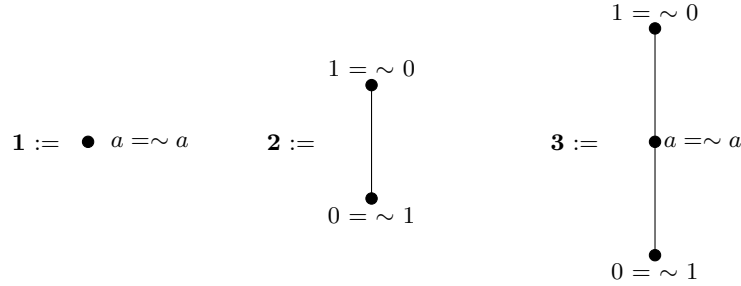
*Proof.* Let us only prove the Kleene property for  $\sim$ . Let  $(a, b), (c, d) \in B^{[2]}$ .  
 $(a, b) \wedge \sim(a, b) = (a, b) \wedge (b^c, a^c) = (a \wedge b^c, b \wedge a^c) = (0, b \wedge a^c).$   
 $(c, d) \vee \sim(c, d) = (c, d) \vee (d^c, c^c) = (c \vee d^c, d \vee c^c) = (c \vee d^c, 1).$   
Hence  $(a, b) \wedge \sim(a, b) \leq (c, d) \vee \sim(c, d).$

In this section we prove the following representation result.

**Theorem 2.** Given a Kleene algebra  $\mathcal{K}$ , there exists a Boolean algebra  $\mathcal{B}_{\mathcal{K}}$  such that  $\mathcal{K}$  can be embedded into  $\mathcal{B}_{\mathcal{K}}^{[2]}$ .

Observe that we already have the following well-known representation theorem, due to the fact that **1**, **2** and **3** (Figure 1) are the only subdirectly irreducible (Kleene) algebras in the variety of Kleene algebras.

**Theorem 3.** [18] Let  $\mathcal{K}$  be a Kleene algebra. There exists a (index) set  $I$  such that  $\mathcal{K}$  can be embedded into  $\mathbf{3}^I$ .



**Fig. 1.** Subdirectly irreducible Kleene algebras

So, to prove the Theorem 2, we prove the following.

**Theorem 4.** For the Kleene algebra  $\mathbf{3}^I$  corresponding to any index set  $I$ , there exists a Boolean algebra  $\mathcal{B}_{\mathbf{3}^I}$  such that  $\mathbf{3}^I \cong (\mathcal{B}_{\mathbf{3}^I})^{[2]}$ .

## 2.1 Completely join irreducible elements of $\mathbf{3}^I$ and $(\mathbf{2}^I)^{[2]}$

Completely join irreducible elements play a fundamental role in establishing isomorphisms between lattice-based algebras. Let us put the basic definitions and notations in place.

**Definition 2.** Let  $\mathcal{L} := (L, \vee, \wedge, 0, 1)$  be a complete lattice.

(i) An element  $a \in L$  is said to be completely join irreducible, when  $a = \vee a_i$  implies that  $a = a_i$  for some  $i$ .

**Notation 1** Let  $\mathcal{J}_L$  denote the set of all completely join irreducible elements of  $L$ , and  $J(x) := \{a \in \mathcal{J}_L : a \leq x\}$ , for any  $x \in L$ .

(ii) A set  $S$  is said to be join dense in  $\mathcal{L}$ , provided every element of  $L$  is the join of some elements from  $S$ .

Observe that the element 0 is always completely join irreducible.

For a given index set  $I$ , let us characterize the sets of completely join irreducible elements of the Kleene algebras  $\mathbf{3}^I$  and  $(\mathbf{2}^I)^{[2]}$ , and prove their join density in the respective lattices. Let  $i, k \in I$ . Denote by  $f_i^x$ ,  $x \in \mathbf{3} := \{0, a, 1\}$ , the following element in  $\mathbf{3}^I$ .

$$f_i^x(k) := \begin{cases} x & \text{if } k = i \\ 0 & \text{otherwise} \end{cases}$$

**Proposition 2.** The set of completely join irreducible elements of  $\mathbf{3}^I$  is given by:

$$\mathcal{J}_{\mathbf{3}^I} = \{f_i^x : i \in I, x \in \mathbf{3}\}.$$

Moreover,  $\mathcal{J}_{\mathbf{3}^I}$  is join dense in  $\mathbf{3}^I$ .

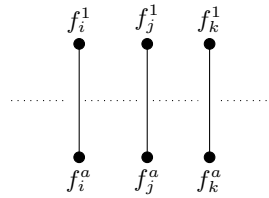
*Proof.* We only consider the non-zero elements. Let  $f_i^a = \vee_{k \in K} f_k$ ,  $K \subseteq I$ . This implies that  $f_i^a(j) = \vee_{k \in K} f_k(j)$ , for each  $j \in I$ . If  $j \neq i$ , by the definition of  $f_i^a$ ,  $f_i^a(j) = 0$ . So  $\vee_{k \in K} f_k(j) = 0$ , whence  $f_k(j) = 0$ , for each  $k \in K$ . If  $j = i$ , then  $f_i^a(j) = a$ , which means  $\vee_{k \in K} f_k(j) = a$ . But as  $a$  is join irreducible in  $\mathbf{3}$ , there exists a  $k' \in K$  such that  $f_{k'}(j) = a$ . Hence  $f_i^a = f_{k'}$ . A similar argument works for  $f_i^1$ .

Now let  $f(\neq 0) \in \mathbf{3}^I$ . Take  $K = I$ , and for each  $j \in I$ , define the element  $f_j$  of  $\mathbf{3}^I$  as

$$f_j(k) := \begin{cases} f(j) & \text{if } k = j \\ 0 & \text{otherwise} \end{cases}$$

Clearly, we have  $f = \vee_{j \in I} f_j$ , where  $f_j \in \mathcal{J}_{\mathbf{3}^I}$ . □

Let us note that for each  $i, j \in I$ ,  $f_i^a \leq f_i^1$ , and if  $i \neq j$ , neither  $f_i^x \leq f_j^y$  nor  $f_j^x \leq f_i^y$  holds for  $x, y \in \{a, 1\}$ . The order structure of the non-zero elements in  $\mathcal{J}_{\mathbf{3}^I}$  can be visualized by Figure 2:



**Fig. 2.** Hasse diagram of  $\mathcal{J}_{\mathbf{3}^I}$

*Example 1.* Let us consider the Kleene algebra  $\mathbf{3}^3$ . The set  $\mathcal{J}_{\mathbf{3}^3}$  of completely join irreducible elements of  $\mathbf{3}^3$  is then given by

$$\mathcal{J}_{\mathbf{3}^3} = \{(0, 0, 0), f_1^a := (a, 0, 0), f_1^1 := (1, 0, 0), f_2^a := (0, a, 0), f_2^1 := (0, 1, 0), f_3^a := (0, 0, a), f_3^1 := (0, 0, 1)\}.$$

Let  $f := (0, a, 1) \in \mathbf{3}^3$ . Then  $f = f_1 \vee f_2 \vee f_3$ , where  $f_1 = (0, 0, 0)$ ,  $f_2 = (0, a, 0)$  and  $f_3 = (0, 0, 1)$ .

As any complete atomic Boolean algebra is isomorphic to  $\mathbf{2}^I$  for some index set  $I$ , henceforth, we shall identify any complete atomic Boolean algebra  $B$  with  $\mathbf{2}^I$ . Now, for any such algebra,  $B^{[2]}$  is a Kleene algebra (cf. Proposition 1); in fact, it is a completely distributive Kleene algebra.

**Proposition 3.** *Let  $B$  be a complete atomic Boolean algebra. The set of completely join irreducible elements of  $B^{[2]}$  is given by*

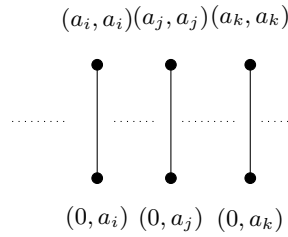
$$\mathcal{J}_{B^{[2]}} = \{(0, a), (a, a) : a \in \mathcal{J}_B\}.$$

*Moreover,  $\mathcal{J}_{B^{[2]}}$  is join dense in  $B^{[2]}$ .*

*Proof.* Let  $a \in \mathcal{J}_B$  and let  $(a, a) = \bigvee_{k \in K} (x_k, y_k)$ ,  $K \subseteq I$ , where  $(x_k, y_k) \in B^{[2]}$  for each  $k \in K$ .  $(a, a) = \bigvee_{k \in K} (x_k, y_k)$  implies  $a = \bigvee_{k \in K} x_k$ . As  $a \in \mathcal{J}_B$ ,  $a = x_{k'}$  for some  $k' \in K$ . We already have  $x_{k'} \leq y_{k'} \leq a$ , hence combining with  $a = x_{k'}$ , we get  $(a, a) = (x_{k'}, y_{k'})$ . With similar arguments one can show that for each  $a \in \mathcal{J}_B$ ,  $(0, a)$  is completely join irreducible.

Now, let  $(x, y) (\neq (0, 0)) \in B^{[2]}$ . Consider the sets  $J(x)$ ,  $J(y)$  (cf. Notation 1, Definition 2). Then  $(x, y) = \bigvee_{a \in J(x)} (a, a) \vee \bigvee_{b \in J(y)} (0, b)$ . Hence  $\mathcal{J}_{B^{[2]}}$  is join dense in  $B^{[2]}$ .  $\square$

For  $a, b (\neq 0) \in \mathcal{J}_B$ ,  $(0, a) \leq (a, a)$ , and if  $a \neq b$ ,  $x, y \in \{a, b\}$  with  $x \neq y$ , neither  $(0, x) \leq (0, y), (y, y)$  nor  $(x, x) \leq (0, y), (y, y)$  holds. Then, similar to the case of  $\mathbf{3}^I$ , the completely join irreducible non-zero elements of  $B^{[2]}$  can be visualized by Figure 3:



**Fig. 3.** Hasse diagram of  $\mathcal{J}_{B^{[2]}}$

*Example 2.* Consider the Boolean algebra  $\mathbf{4}$  of four elements with atoms  $a$  and  $b$ . The set of completely join irreducible elements of  $\mathbf{4}^{[2]}$  is given by  $\mathcal{J}_{\mathbf{4}^{[2]}} = \{(0, 0), (0, a), (a, a), (0, b), (b, b)\}$ .

Let  $(a, 1) \in \mathbf{4}^{[2]}$ . Then  $J(a) = \{0, a\}$  and  $J(1) = \{0, a, b\}$ . Hence  $(a, 1) = (0, 0) \vee (a, a) \vee (0, a) \vee (0, b)$ .

## 2.2 Structural theorem for Kleene algebras

Let us first present the basic lattice-theoretic definitions and results that will be required to arrive at the proof of Theorem 4.

### Definition 3.

1. (a) A complete lattice of sets is a family  $\mathcal{F}$  such that  $\bigcup \mathcal{H}$  and  $\bigcap \mathcal{H}$  belong to  $\mathcal{F}$  for any  $\mathcal{H} \subseteq \mathcal{F}$ .
2. Let  $L$  be a complete lattice.
  - (a)  $L$  is said to be algebraic if any element  $x \in L$  is the join of a set of compact elements of  $L$ .
  - (b)  $L$  is said to satisfy the Join-Infinite Distributive Law, if for any subset  $\{y_j\}_{j \in J}$  of  $L$  and any  $x \in L$ ,

$$(JID) \quad x \wedge \bigvee_{j \in J} y_j = \bigvee_{j \in J} x \wedge y_j.$$

**Theorem 5.** [15] Let  $L$  be a lattice. The following are equivalent.

1.  $L$  is complete, satisfies (JID) and the set of completely join irreducible elements is join dense in  $L$ .
2.  $L$  is completely distributive and algebraic.

It can be easily seen that both the lattices  $\mathbf{3}^I$  and  $(\mathbf{2}^I)^{[2]}$  are complete and satisfy (JID). We have already observed from Section 2.1 that the sets of completely join irreducible elements of  $\mathbf{3}^I$  and  $(\mathbf{2}^I)^{[2]}$  are join dense in the respective lattices. So Theorem 5(1) holds for  $\mathbf{3}^I$  and  $(\mathbf{2}^I)^{[2]}$ , and therefore,  $\mathbf{3}^I$  and  $(\mathbf{2}^I)^{[2]}$  are completely distributive and algebraic lattices.

*Note 1.* For the remaining study, let us fix an index set  $I$ . In the rest of our paper, we exclude 0 from the list of completely join irreducible elements and the lattice in the definition of join density, as this does not change the results.

Let us recall from Section 2.1 that the completely join irreducible elements of  $\mathbf{3}^I$  and  $(\mathbf{2}^I)^{[2]}$  are given by  $\mathcal{J}_{\mathbf{3}^I} = \{f_i^a, f_i^1 : i \in I\}$  and  $\mathcal{J}_{(\mathbf{2}^I)^{[2]}} = \{(0, g_i^1), (g_i^1, g_i^1) : i \in I\}$ , where  $g_i^1$ 's are the atoms or non-zero completely join irreducible elements of the Boolean algebra  $\mathbf{2}^I$ , defined as the earlier  $f_i^1$ , with domain restricted to  $\mathbf{2}$ .

$$g_i^1(k) := \begin{cases} 1 & \text{if } k = i \\ 0 & \text{otherwise} \end{cases}$$

**Theorem 6.** The sets of completely join irreducible elements of  $\mathbf{3}^I$  and  $(\mathbf{2}^I)^{[2]}$  are order isomorphic.

*Proof.* We define the map  $\phi : \mathcal{J}_{\mathbf{3}^I} \rightarrow \mathcal{J}_{(\mathbf{2}^I)^{[2]}}$  as follows. For  $i \in I$ ,

$$\begin{aligned}\phi(f_i^a) &:= (0, g_i^1), \\ \phi(f_i^1) &:= (g_i^1, g_i^1).\end{aligned}$$

One can show that  $\phi$  is an order isomorphism.

- $f_i^x \leq f_j^y$  if and only if  $i = j$  and  $x, y = a$  or  $x, y = 1$ , or  $x = a, y = 1$ . In any case, by definition of  $\phi$ ,  $\phi(f_i^x) \leq \phi(f_j^y)$ .
- Let  $\phi(f_i^x) \leq \phi(f_j^y)$  and assume  $\phi(f_i^x) = (g_k^1, g_l^1)$  and  $\phi(f_j^y) = (g_m^1, g_n^1)$ . But then again:  $k = l = m = n$  or  $g_k^1 = g_m^1 = 0, l = n$  or  $g_k^1 = 0, l = m = n$ . Again, following the definition of  $\phi$ , we have  $f_i^x \leq f_j^y$ .
- If  $(0, g_i^1) \in \mathcal{J}_{(\mathbf{2}^I)^{[2]}}$ , then  $\phi(f_i^a) = (0, g_i^1)$ . Similarly for  $(g_i^1, g_i^1)$ . Hence  $\phi$  is onto.  $\square$

**Lemma 1.** [19] *Let  $L$  and  $K$  be two completely distributive lattices. Further, let  $\mathcal{J}_L$  and  $\mathcal{J}_K$  be join dense in  $L$  and  $K$  respectively. Let  $\phi : \mathcal{J}_L \rightarrow \mathcal{J}_K$  be an order isomorphism. Then the extension map  $\Phi : L \rightarrow K$  given by*

$$\Phi(x) := \bigvee (\phi(J(x))) \text{ (where } J(x) := \{a \in \mathcal{J}_L : a \leq x\}), x \in L,$$

*is a lattice isomorphism.*

Using Theorem 6 and Lemma 1, we have the following:

**Theorem 7.** *The algebras  $\mathbf{3}^I$  and  $(\mathbf{2}^I)^{[2]}$  are lattice isomorphic.*

Now, in order to obtain Theorem 4, we would like to extend the above lattice isomorphism to a Kleene isomorphism. We use the technique of Järvinen in [13]. Let us present the preliminaries.

Let  $\mathbb{K} := (K, \vee, \wedge, \sim, 0, 1)$  be a completely distributive De Morgan algebra. Define for any  $j \in \mathcal{J}_K$ ,

$$j^* := \bigwedge \{x \in K : x \not\leq \sim j\}.$$

Then  $j^* \in \mathcal{J}_K$ . For complete details on  $j^*$ , one may refer to [13]. Further, it is shown that Lemma 1 can be extended to De Morgan algebras defined over algebraic lattices.

**Theorem 8.** *Let  $\mathbb{L} := (L, \vee, \wedge, \sim, 0, 1)$  and  $\mathbb{K} := (K, \vee, \wedge, \sim, 0, 1)$  be two De Morgan algebras defined on algebraic lattices. If  $\phi : \mathcal{J}_L \rightarrow \mathcal{J}_K$  is an order isomorphism such that*

$$\phi(j^*) = \phi(j)^*, \text{ for all } j \in \mathcal{J}_L,$$

*then  $\Phi$  is an isomorphism between the algebras  $\mathbb{L}$  and  $\mathbb{K}$ .*

Let  $f_i^a \in \mathcal{J}_{\mathbf{3}^I}$ . Then by definition,  $(f_i^a)^* = \bigwedge \{f \in \mathbf{3}^I : f \not\leq \sim (f_i^a)\}$ , where for each  $i \in I$ ,

$$\sim (f_i^a)(k) = \begin{cases} a & \text{if } k = i \\ 1 & \text{otherwise} \end{cases}$$

Clearly, we have  $f_i^1 \not\leq \sim (f_i^a)$ . Now let  $f \not\leq \sim (f_i^a)$ . Then what does  $f$  look like? If  $k \neq i$ ,  $f(k) \leq \sim (f_i^a)(k) = 1$ . So, for  $f \not\leq \sim (f_i^a)$ ,  $f(i)$  has to be 1 (otherwise  $f(i) = 0, a$  will lead to  $f \leq \sim (f_i^a)$ ). Hence,  $f_i^1 \leq f$  and  $(f_i^a)^* = f_i^1$ . Similarly, one can easily show that  $(f_i^1)^* = f_i^a$ .



On the other hand, let us consider  $(0, g_i^1) \in \mathcal{J}_{(\mathbf{2}^I)^{[2]}}$ . Then,  $(0, g_i^1)^* = \bigwedge \{(g, g') \in (\mathbf{2}^I)^{[2]} : (g, g') \not\sim (0, g_i^1)\}$ . By definition of  $\sim$ , we have  $\sim (0, g_i^1) = ((g_i^1)^c, 0^c) = ((g_i^1)^c, 1)$ . Observe that  $(g_i^1, g_i^1) \not\leq ((g_i^1)^c, 1)$ , as,  $g_i^1 \not\leq (g_i^1)^c$  is true in a Boolean algebra. Now, let  $(g, g') \in \mathcal{J}_{(\mathbf{2}^I)^{[2]}}$  be such that  $(g, g') \not\sim (0, g_i^1) = ((g_i^1)^c, 1)$ . But we have  $g' \leq 1$ , so for  $(g, g') \not\sim (0, g_i^1)$  to hold, we must have  $g \not\leq (g_i^1)^c$ .  $g_i^1$  is an atom of  $\mathbf{2}^I$  and  $g \not\leq (g_i^1)^c$  imply  $g_i^1 \leq g$ . Hence  $(g_i^1, g_i^1) \leq (g, g')$ , and we get  $(0, g_i^1)^* = (g_i^1, g_i^1)$ . Similarly, we have  $(g_i^1, g_i^1)^* = (0, g_i^1)$ . Let us summarize these observations in the following lemma.

**Lemma 2.** *The completely distributive De Morgan algebra  $\mathbf{3}^I$  has the following properties. For each  $i \in I$ ,  $f_i^a, f_i^1 \in \mathcal{J}_{\mathbf{3}^I}$  and  $(0, g_i^1), (g_i^1, g_i^1) \in \mathcal{J}_{(\mathbf{2}^I)^{[2]}}$  we have,*

1.  $(f_i^a)^* = f_i^1, (0, g_i^1)^* = (g_i^1, g_i^1)$ .
2.  $(f_i^1)^* = f_i^a, (g_i^1, g_i^1)^* = (0, g_i^1)$ .

Now we return to Theorem 4.

**Proof of Theorem 4:**

Let the Kleene algebra  $\mathbf{3}^I$  be given. Consider  $\mathbf{2}^I$  as a Boolean subalgebra of  $\mathbf{3}^I$ . Using the definition of  $\phi$  (cf. Theorem 6) and its extension (cf. Lemma 1), and using Lemma 2 we have, for each  $i \in I$ ,

$$\phi((f_i^a)^*) = \phi(f_i^1) = (g_i^1, g_i^1) = \phi(f_i^a)^*, \phi((f_i^1)^*) = \phi(f_i^a) = (0, g_i^1) = \phi(f_i^1)^*.$$

By Theorem 7,  $\phi$  is an order isomorphism between  $\mathcal{J}_{\mathbf{3}^I}$  and  $\mathcal{J}_{(\mathbf{2}^I)^{[2]}}$ . Hence using Theorem 8,  $\Phi$  is an isomorphism between the De Morgan algebras  $\mathbf{3}^I$  and  $(\mathbf{2}^I)^{[2]}$ . As both the algebras are also Kleene algebras, which are also equational algebras defined over De Morgan algebras, the De Morgan isomorphism  $\Phi$  extends to Kleene isomorphism.  $\square$

Let us illustrate the above theorem through examples.

*Example 3.* Consider the Kleene algebra  $\mathbf{3} := \{0, a, 1\}$ . Then  $\mathcal{J}_{\mathbf{3}} = \{a, 1\}$ . For  $\mathbf{2} := \{0, 1\}$ ,  $\mathbf{2}^{[2]} = \{(0, 0), (0, 1), (1, 1)\}$  and  $\mathcal{J}_{\mathbf{2}^{[2]}} = \{(0, 1), (1, 1)\}$ . Further,  $a^* = 1$ ,  $1^* = a$  and  $(0, 1)^* = (1, 1)$  and  $(1, 1)^* = (0, 1)$ .

Then  $\phi : \mathcal{J}_{\mathbf{3}} \rightarrow \mathcal{J}_{\mathbf{2}^{[2]}}$  is defined as

$$\begin{aligned} \phi(a) &:= (0, 1), \\ \phi(1) &:= (1, 1). \end{aligned}$$

Hence the extension map  $\Phi : \mathbf{3} \rightarrow \mathbf{2}^{[2]}$  is given as

$$\begin{aligned} \Phi(a) &:= (0, 1), \\ \Phi(1) &:= (1, 1), \\ \Phi(0) &:= (0, 0). \end{aligned}$$

The diagrammatic illustration of this example is given in Figure 4.

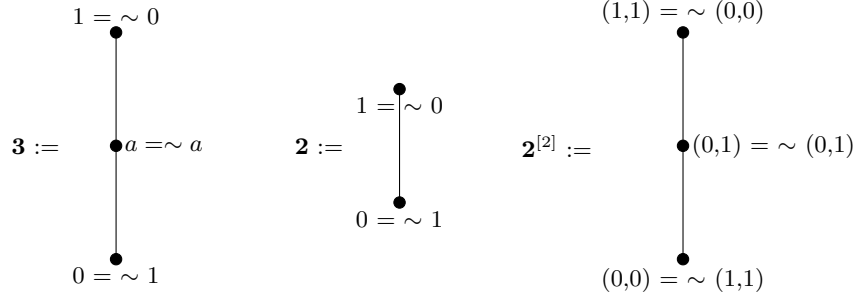
*Example 4.* Now, let us consider the Kleene algebra  $\mathbf{3} \times \mathbf{3}$ .

$\mathbf{3} \times \mathbf{3} := \{(0, 0), (0, a), (0, 1), (a, 1), (1, 1), (a, 0), (1, 0), (1, a), (a, a)\}$ .

$\mathcal{J}_{\mathbf{3} \times \mathbf{3}} = \{(0, a), (0, 1), (a, 0), (1, 0)\}$  and

$(0, a)^* = (0, 1), (0, 1)^* = (0, a), (a, 0)^* = (1, 0), (1, 0)^* = (a, 0)$ .

Consider the Boolean subalgebra  $\mathbf{2} \times \mathbf{2} = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$  of  $\mathbf{3} \times \mathbf{3}$ .

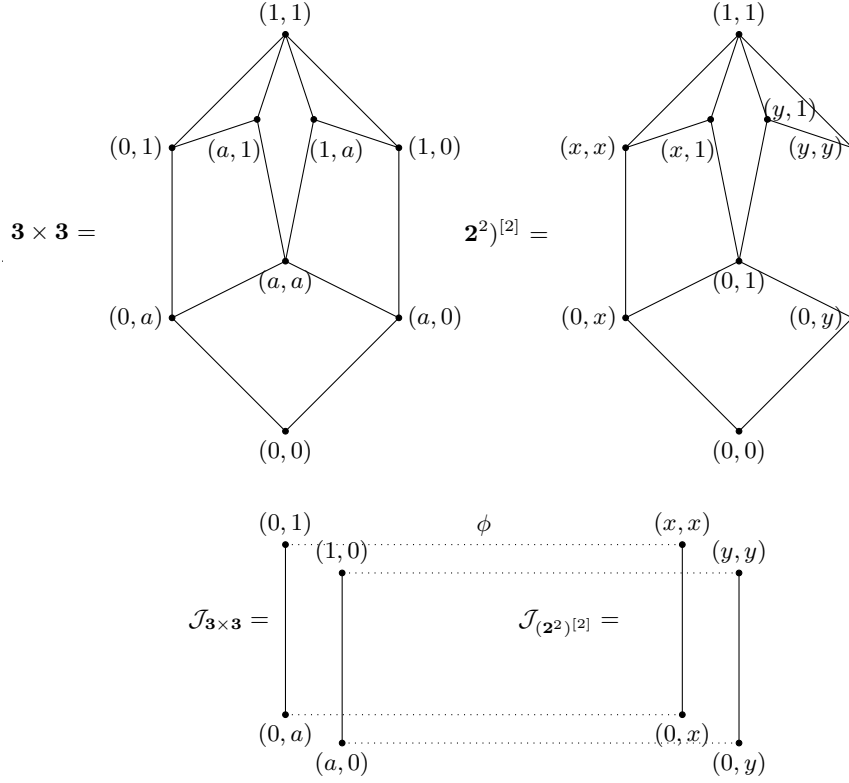
**Fig. 4.**  $\mathbf{3} \cong \mathbf{2}^{[2]}$ 

For convenience, let us change the notations. We represent the set  $\mathbf{2} \times \mathbf{2}$  and its elements as  $\mathbf{2}^2 = \{0, x, y, 1\}$ , where  $(0, 0)$  is replaced by 0,  $(0, 1)$  is replaced by  $x$ ,  $(1, 0)$  is replaced by  $y$ , and  $(1, 1)$  is replaced by 1. Then  $(\mathbf{2}^2)^{[2]} = \{(0, 0), (0, x), (0, 1), (0, y), (x, x), (x, 1), (y, 1), (y, y), (1, 1)\}$ , and  $\mathcal{J}_{(\mathbf{2}^2)^{[2]}} = \{(0, x), (0, y), (x, x), (y, y)\}$ . Further,  $(0, x)^* = (x, x)$ ,  $(x, x)^* = (0, x)$  and  $(0, y)^* = (y, y)$ ,  $(y, y)^* = (0, y)$ . The diagrammatic illustration of the isomorphism between  $\mathbf{3} \times \mathbf{3}$  and  $(\mathbf{2}^2)^{[2]}$  is given in Figure 5.

### 3 The logic $\mathcal{L}_K$ for Kleene algebras and a 3-valued semantics

As mentioned earlier, Moisil in 1941 (cf. [7]) proved that  $B^{[2]}$  forms a 3-valued LM algebra. Varlet (cf. [9]) noted the equivalence between regular double Stone algebras and 3-valued LM algebras, whence  $B^{[2]}$  can be given the structure of a regular double Stone algebra as well. So, while discussing the logic corresponding to the structures  $B^{[2]}$ , one is naturally led to 3-valued Łukasiewicz logic. Here, due to Proposition 1 and Theorem 2, we focus on  $B^{[2]}$  as a *Kleene algebra*, and study the (propositional) logic corresponding to the class of Kleene algebras and the structures  $B^{[2]}$ . We denote this system as  $\mathcal{L}_K$ , and present it below.

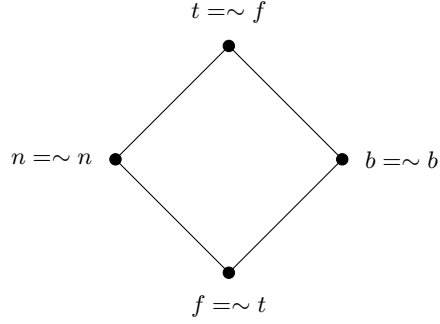
Our approach to the study is motivated by Dunn's 4-valued semantics of the De Morgan consequence system [6]. The 4-valued semantics arises from the fact that each element of a De Morgan algebra can be looked upon as a pair of sets. Now, using Stone's representation, each Boolean algebra is embeddable in a power set algebra, whence  $B^{[2]}$ , for any Boolean algebra  $B$ , is embeddable in  $\mathcal{P}(U)^{[2]}$ , for some set  $U$ . Thus, because of Theorem 2, one can say that each element of a Kleene algebra can also be looked upon as a pair of sets. As shown in Example 3 above, the Kleene algebra  $\mathbf{3} \cong \mathbf{2}^{[2]}$ . We exploit the fact that  $\mathbf{3}$ , in particular, can be represented as a Kleene algebra of pairs of sets, to get completeness of the logic  $\mathcal{L}_K$  for Kleene algebras with respect to a 3-valued semantics.



**Fig. 5.**  $\mathbf{3} \times \mathbf{3} \cong (\mathbf{2}^2)^{[2]}$

The *Kleene* axiom  $\alpha \wedge \sim \alpha \vdash \beta \vee \sim \beta$ , given by Kalman [1], was studied by Dunn [6, 20] in the context of providing a 3-valued semantics for a fragment of relevance logic. He showed that the De Morgan consequence system coupled with the Kleene axiom (the resulting consequence relation being denoted as  $\vdash_{Kalman}$ ), is sound and complete with respect to a semantic consequence relation (denoted  $\models_{0,1}^{\mathbf{3}_R}$ ) defined on  $\mathbf{3}_R$ , the *right hand chain* of the De Morgan lattice  $\mathbf{4}$  given in Figure 6.  $\mathbf{3}_R$  is the side of  $\mathbf{4}$  in which the elements are interpreted as  $t$ (rue),  $f$ (alse) and  $b$ (oth), and  $\models_{0,1}^{\mathbf{3}_R}$  essentially incorporates truth *and* falsity preservation by valuations in its definition. He called this consequence system, the *Kalman consequence system*. The completeness result for the Kalman consequence system is obtained considering all 4-valued valuations restricted to  $\mathbf{3}_R$ : the proof makes explicit reference to valuations on  $\mathbf{4}$ .

The logic  $\mathcal{L}_K$  ( $K$  for Kalman and Kleene) that we are considering in our work, has a consequence system that is  $\vdash_{Kalman}$ , with slight modifications.  $\mathcal{L}_K$  is shown to be sound and complete with respect to a 3-valued semantics that is based on the same idea underlying the consequence relation  $\models_{0,1}^{\mathbf{3}_R}$ , viz. that of

**Fig. 6.** De Morgan lattice 4

truth as well as falsity preservation. However, the definitions and proofs in this case, *do not refer to 4*.

Let us present  $\mathcal{L}_K$ . The language consists of

- propositional variables:  $p, q, r, \dots$
- propositional constants:  $\top, \perp$ .
- logical connectives:  $\vee, \wedge, \sim$ .

The well-formed formulae of the logic are defined through the scheme:

$$\top \mid \perp \mid p \mid \alpha \vee \beta \mid \alpha \wedge \beta \mid \sim \alpha.$$

**Notation 2** Denote the set of well-formed formulae by  $\mathcal{F}$ .

The consequence relation  $\vdash_{\mathcal{L}_K}$  is now given through the following postulates and rules, taken from [6] and [14]. These define reflexivity and transitivity of  $\vdash$ , introduction, elimination principles and the distributive law for the connectives  $\wedge$  and  $\vee$ , contraposition and double negation laws for the negation operator  $\sim$ , the Kleene property for  $\sim$ , and some basic requirements from the propositional constants  $\top, \perp$ . Let  $\alpha, \beta, \gamma \in \mathcal{F}$ .

**Definition 4.** ( $\mathcal{L}_K$ - postulates)

1.  $\alpha \vdash \alpha$
2.  $\alpha \vdash \beta, \beta \vdash \gamma \mid \alpha \vdash \gamma$ .
3.  $\alpha \wedge \beta \vdash \alpha, \alpha \wedge \beta \vdash \beta$ .
4.  $\alpha \vdash \beta, \alpha \vdash \gamma \mid \alpha \vdash \beta \wedge \gamma$ .
5.  $\alpha \vdash \gamma, \beta \vdash \gamma \mid \alpha \vee \beta \vdash \gamma$ .
6.  $\alpha \vdash \alpha \vee \beta, \beta \vdash \alpha \vee \beta$ .
7.  $\alpha \wedge (\beta \vee \gamma) \vdash (\alpha \vee \beta) \wedge (\alpha \vee \gamma)$  (Distributivity).
8.  $\alpha \vdash \beta \mid \sim \beta \vdash \sim \alpha$  (Contraposition).
9.  $\sim \alpha \wedge \sim \beta \vdash \sim (\alpha \vee \beta)$  ( $\vee$ -linearity).
10.  $\alpha \vdash \top$  (Top).
11.  $\perp \vdash \alpha$  (Bottom).

12.  $\top \vdash \sim \perp$  (Nor).
13.  $\alpha \vdash \sim \sim \alpha$ .
14.  $\sim \sim \alpha \vdash \alpha$ .
15.  $\alpha \wedge \sim \alpha \vdash \beta \vee \sim \beta$  (Kalman/Kleene).

Let us now consider any Kleene algebra  $(K, \vee, \wedge, \sim, 0, 1)$ . We first define valuations on  $K$ .

**Definition 5.** A map  $v : \mathcal{F} \rightarrow K$  is called a valuation on  $K$ , if it satisfies the following properties for any  $\alpha, \beta \in \mathcal{F}$ .

1.  $v(\alpha \vee \beta) = v(\alpha) \vee v(\beta)$ .
2.  $v(\alpha \wedge \beta) = v(\alpha) \wedge v(\beta)$ .
3.  $v(\sim \alpha) = \sim v(\alpha)$ .
4.  $v(\perp) = 0$ .
5.  $v(\top) = 1$ .

A consequent  $\alpha \vdash \beta$  is *valid in  $K$  under the valuation  $v$* , if  $v(\alpha) \leq v(\beta)$ . If the consequent is valid under all valuations on  $K$ , then it is *valid in  $K$* . Let  $\mathcal{A}$  be a class of Kleene algebras. If the consequent  $\alpha \vdash \beta$  is valid in each algebra of  $\mathcal{A}$ , then we say  $\alpha \vdash \beta$  is *valid in  $\mathcal{A}$* , and denote it as  $\alpha \models_{\mathcal{A}} \beta$ .

Let  $\mathcal{A}_K$  denote the class of *all* Kleene algebras. We have, in the classical manner,

**Theorem 9.**  $\alpha \vdash_{\mathcal{L}_K} \beta$  if and only if  $\alpha \models_{\mathcal{A}_K} \beta$ , for any  $\alpha, \beta \in \mathcal{F}$ .

Let us now focus on valuations on the Kleene algebra  $B^{[2]}$ . Then for  $\alpha \in \mathcal{F}$ ,  $v(\alpha)$  is a pair of the form  $(a, b)$ . Suppose for  $\beta \in \mathcal{F}$ ,  $v(\beta) := (c, d)$ . By definition, the consequent  $\alpha \vdash \beta$  is valid in  $B^{[2]}$  under  $v$ , when  $v(\alpha) \leq v(\beta)$ , i.e.,  $(a, b) \leq (c, d)$ , or  $a \leq c$  and  $b \leq d$ .

Let  $\mathcal{A}_{KB^{[2]}}$  denote the class of Kleene algebras formed by the sets  $B^{[2]}$ , for *all* Boolean algebras  $B$ . Then we have

**Theorem 10.**  $\alpha \models_{\mathcal{A}_K} \beta$  if and only if  $\alpha \models_{\mathcal{A}_{KB^{[2]}}} \beta$ , for any  $\alpha, \beta \in \mathcal{F}$ .

*Proof.* Let  $\alpha \models_{\mathcal{A}_{KB^{[2]}}} \beta$ . Consider any Kleene algebra  $(K, \vee, \wedge, \sim, 0, 1)$ , and let  $v$  be a valuation on  $K$ . By Theorem 2, there exists a Boolean algebra  $B$  such that  $K$  is embedded in  $B^{[2]}$ . Let  $\phi$  denote the embedding. It is a routine verification that  $\phi \circ v$  is a valuation on  $B^{[2]}$ . The other direction is trivial, as  $\mathcal{A}_{KB^{[2]}}$  is a subclass of  $\mathcal{A}_K$ .  $\square$

On the other hand, as observed earlier, the structure  $B^{[2]}$  is embeddable in  $\mathcal{P}(U)^{[2]}$  for some set  $U$ , utilizing Stone's representation. Hence if  $v$  is a valuation on  $B^{[2]}$ , it can be extended to a valuation on  $\mathcal{P}(U)^{[2]}$ . Let  $\mathcal{A}_{K\mathcal{P}(U)^{[2]}}$  denote the class of Kleene algebras of the form  $\mathcal{P}(U)^{[2]}$ , for *all* sets  $U$ . So, we get from Theorem 10 the following.

**Corollary 1.**  $\alpha \models_{\mathcal{A}_K} \beta$  if and only if  $\alpha \models_{\mathcal{A}_{K\mathcal{P}(U)^{[2]}}} \beta$ , for any  $\alpha, \beta \in \mathcal{F}$ .

Following [6], we now consider semantic consequence relations defined by valuations  $v : \mathcal{F} \rightarrow \mathbf{3}$  on the Kleene algebra  $\mathbf{3}$ . Let us re-label the elements of  $\mathbf{3}$  as  $f, u, t$ , giving the standard truth value connotations.

**Definition 6.** Let  $\alpha, \beta \in \mathcal{F}$ .

- $\alpha \models_t \beta$  if and only if, if  $v(\alpha) = t$  then  $v(\beta) = t$  (Truth preservation).
- $\alpha \models_f \beta$  if and only if, if  $v(\beta) = f$  then  $v(\alpha) = f$  (Falsity preservation).
- $\alpha \models_{t,f} \beta$  if and only if,  $\alpha \models_t \beta$  and  $\alpha \models_f \beta$ .

We adopt  $\models_{t,f}$  as the semantic consequence relation for the logic  $\mathcal{L}_K$ . Note that the consequence relation  $\models_t$  is the consequence relation used in [21] to interpret the strong Kleene logic. In case of Dunn's 4-valued semantics, the consequence relations  $\models_t, \models_f$  and  $\models_{t,f}$  are defined using valuations on  $\mathbf{4}$ . As shown in [6], all the three turn out to be equivalent. In order to capture the first-degree entailment fragment of relevance logic, Dunn subsequently uses the semantic consequence relation  $\models_{0,1}^{\mathbf{3}_R}$ , defined by valuations restricted to  $\mathbf{3}_R$ , the right hand chain of  $\mathbf{4}$ . Observe that for valuations on  $\mathbf{3}$  that are being considered here, the consequence relations  $\models_t, \models_f$  and  $\models_{t,f}$  are not equivalent:  $\alpha \wedge \sim \alpha \models_t \beta$ , but  $\alpha \wedge \sim \alpha \not\models_f \beta$ ;  $\beta \models_f \alpha \vee \sim \alpha$ , but  $\beta \not\models_t \alpha \vee \sim \alpha$ .

**Theorem 11.**  $\alpha \models_{\mathcal{A}_{K\mathcal{P}(U)^{[2]}}} \beta$  if and only if  $\alpha \models_{t,f} \beta$ , for any  $\alpha, \beta \in \mathcal{F}$ .

*Proof.* Let  $\alpha \models_{\mathcal{A}_{K\mathcal{P}(U)^{[2]}}} \beta$ , and  $v : \mathcal{F} \rightarrow \mathbf{3}$  be a valuation. As we have already noted,  $\mathbf{3} \cong \mathcal{P}(U)^{[2]}$ . If the correspondence is denoted by  $\phi$ ,  $\phi \circ v$  is a valuation on  $\mathcal{P}(U)^{[2]}$ . Then  $(\phi \circ v)(\alpha) \leq (\phi \circ v)(\beta)$  implies  $v(\alpha) \leq v(\beta)$ . Thus if  $v(\alpha) = t$ , we have  $v(\beta) = t$ , and if  $v(\beta) = f$ , then also  $v(\alpha) = f$ .

Now let  $\alpha \models_{t,f} \beta$ . Let  $U$  be a set, and  $\mathcal{P}(U)^{[2]}$  be the corresponding Kleene algebra. Let  $v$  be a valuation on  $\mathcal{P}(U)^{[2]}$  – we need to show  $v(\alpha) \leq v(\beta)$ . For any  $\gamma \in \mathcal{F}$  with  $v(\gamma) := (A, B)$  and for each  $x \in U$ , define a map  $v_x : \mathcal{F} \rightarrow \mathbf{3}$  as

$$v_x(\gamma) := \begin{cases} t & \text{if } x \in A \\ u & \text{if } x \in B \setminus A \\ f & \text{if } x \notin B. \end{cases}$$

We show that  $v_x$  is a valuation.

Consider any  $\gamma, \delta \in \mathcal{F}$ , with  $v(\gamma) := (A, B)$  and  $v(\delta) := (C, D)$ .

1.  $v_x(\gamma \wedge \delta) = v_x(\gamma) \wedge v_x(\delta)$ .

Note that  $v(\gamma \wedge \delta) = (A \cap C, B \cap D)$ .

Case 1  $v_x(\gamma) = t$  and  $v_x(\delta) = t$ : Then  $x \in A \cap C$ , and we have  $v_x(\gamma \wedge \delta) = t = v_x(\gamma) \wedge v_x(\delta)$ .

Case 2  $v_x(\gamma) = t$  and  $v_x(\delta) = u$ :  $x \in A$ ,  $x \in D$  and  $x \notin C$ , which imply  $x \notin A \cap C$  but  $x \in B \cap D$ . Hence  $v_x(\gamma \wedge \delta) = u = v_x(\gamma) \wedge v_x(\delta)$ .

Case 3  $v_x(\gamma) = t$  and  $v_x(\delta) = f$ :  $x \in A$ ,  $x \notin D$ , which imply  $x \notin B \cap D$ . Hence  $v_x(\gamma \wedge \delta) = f = v_x(\gamma) \wedge v_x(\delta)$ .

Case 4  $v_x(\gamma) = u$  and  $v_x(\delta) = f$ :  $x \notin A$  but  $x \in B$  and  $x \notin D$ , which imply  $x \notin B \cap D$ . Hence  $v_x(\gamma \wedge \delta) = f = v_x(\gamma) \wedge v_x(\delta)$ .

Case 5  $v_x(\gamma) = u$ ,  $v_x(\delta) = u$ :  $x \in B$  but  $x \notin A$  and  $x \in D$  but  $x \notin C$ . So,

$x \in B \cap D$  and  $x \notin A \cap C$ . Hence  $v_x(\gamma \wedge \delta) = u = v_x(\gamma) \wedge v_x(\delta)$ .

Case 6  $v_x(\gamma) = f$ ,  $v_x(\delta) = f$ :  $x \notin B$  and  $x \notin D$ . So,  $x \notin B \cap D$ . Hence  $v_x(\gamma \wedge \delta) = f = v_x(\gamma) \wedge v_x(\delta)$ .

2.  $v_x(\gamma \vee \delta) = v_x(\gamma) \vee v_x(\delta)$ .

Observe that  $v(\gamma \vee \delta) = (A \cup C, B \cup D)$ .

Case 1  $v_x(\gamma) = t$  and  $v_x(\delta) = t$ : Then  $x \in A$ ,  $x \in C$ , which imply  $x \in A \cup C$ . Hence  $v_x(\gamma \vee \delta) = t = v_x(\gamma) \vee v_x(\delta)$ .

Case 2  $v_x(\gamma) = t$  and  $v_x(\delta) = u$ :  $x \in A$ ,  $x \in D$  and  $x \notin C$ , in any way  $x \in A \cup C$ . Hence  $v_x(\gamma \vee \delta) = t = v_x(\gamma) \vee v_x(\delta)$ .

Case 3  $v_x(\gamma) = t$  and  $v_x(\delta) = f$ :  $x \in A$ ,  $x \notin D$ , which imply  $x \in A \cup C$ . Hence  $v_x(\gamma \vee \delta) = t = v_x(\gamma) \vee v_x(\delta)$ .

Case 4  $v_x(\gamma) = u$  and  $v_x(\delta) = f$ :  $x \notin A$  but  $x \in B$  and  $x \notin D$ , which imply  $x \notin A \cup C$  but  $x \in B \cup D$ . Hence  $v_x(\gamma \vee \delta) = u = v_x(\gamma) \vee v_x(\delta)$ .

Case 5  $v_x(\gamma) = u$ ,  $v_x(\delta) = u$ :  $x \in B$  but  $x \notin A$  and  $x \in D$  but  $x \notin C$ . So,  $x \in B \cup D$  and  $x \notin A \cup C$ . Hence  $v_x(\gamma \vee \delta) = u = v_x(\gamma) \vee v_x(\delta)$ .

Case 6  $v_x(\gamma) = f$ ,  $v_x(\delta) = f$ :  $x \notin B$  and  $x \notin D$ . So,  $x \notin B \cup D$ . Hence  $v_x(\gamma \wedge \delta) = f = v_x(\gamma) \wedge v_x(\delta)$ .

3.  $v_x(\sim \gamma) = \sim v_x(\gamma)$ .

Note that  $v(\sim \gamma) = (B^c, A^c)$ .

Case 1  $v_x(\gamma) = t$ : Then  $x \in A$ , i.e.  $x \notin A^c$ . Hence  $v_x(\sim \gamma) = f = \sim v_x(\gamma)$ .

Case 2  $v_x(\gamma) = u$ :  $x \notin A$  but  $x \in B$ . So  $x \in A^c$  and  $x \notin B^c$ . Hence  $v_x(\sim \gamma) = u = \sim v_x(\gamma)$ .

Case 3  $v_x(\gamma) = f$ :  $x \notin B$ , i.e.  $x \in B^c$ . So  $v_x(\sim \gamma) = t = \sim v_x(\gamma)$ .

Hence  $v_x$  is a valuation in **3**. Now let us show that  $v(\alpha) \leq v(\beta)$ . Let  $v(\alpha) := (A', B')$ ,  $v(\beta) := (C', D')$ , and  $x \in A'$ . Then  $v_x(\alpha) = t$ , and as  $\alpha \models_{t,f} \beta$ , by definition,  $v_x(\beta) = t$ . This implies  $x \in C'$ , whence  $A' \subseteq C'$ .

On the other hand, if  $x \notin D'$ ,  $v_x(\beta) = f$ . Hence  $v_x(\alpha) = f$ , so that  $x \notin B'$ , giving  $B' \subseteq D'$ .  $\square$

Note that the above proof cannot be applied on the Kleene algebra  $B^{[2]}$  instead of  $\mathcal{P}(U)^{[2]}$ , as we have used set representations explicitly.

An immediate consequence of Theorem 9, Corollary 1 and Theorem 11 is the following.

**Theorem 12.**  $\alpha \vdash_{\mathcal{L}_K} \beta$  if and only if  $\alpha \models_{t,f} \beta$ , for any  $\alpha, \beta \in \mathcal{F}$ .

## 4 Rough set semantics for $\mathcal{L}_K$

Rough set theory, introduced by Pawlak [10] in 1982, deals with a domain  $U$  (set of objects) and an equivalence (*indiscernibility*) relation  $R$  on  $U$ . The pair  $(U, R)$  is called an (Pawlak) *approximation space*. For any  $A \subseteq U$ , one defines the *lower* and *upper approximations* of  $A$  in the approximation space  $(U, R)$ , denoted  $\mathbf{L}A$  and  $\mathbf{U}A$  respectively, as follows.

$$\begin{aligned} \mathbf{L}A &= \bigcup \{[x] : [x] \subseteq A\}, \\ \mathbf{U}A &= \bigcup \{[x] : [x] \cap A \neq \emptyset\}. \end{aligned} \tag{*}$$

As the information about the objects of the domain is available modulo the equivalence classes in  $U$ , description of any concept, represented extensionally as the subset  $A$  of  $U$ , is inexact, and one ‘approximates’ the description from within and outside, through the lower and upper approximations respectively. Unions of equivalence classes are termed as *definable* sets, signifying exact description in the context of the given information. In particular, sets of the form  $LA$ ,  $UA$  are definable sets.

**Definition 7.** Let  $(U, R)$  be an approximation space. For each  $A \subseteq U$ , the ordered pair  $(LA, UA)$  is called a rough set in  $(U, R)$ .

**Notation 3**  $\mathcal{RS} := \{(LA, UA) : A \subseteq U\}$ .

The ordered pair  $(D_1, D_2)$ , where  $D_1 \subseteq D_2$  and  $D_1, D_2$  are definable sets, is called a generalized rough set in  $(U, R)$ .

**Notation 4**  $\mathcal{D}$  denotes the collection of definable sets and  $\mathcal{R}$  that of the generalized rough sets in  $(U, R)$ .

In the following, we proceed to establish part (ii) of Theorem 1 (cf. Section 1). In Section 4.2, we formalize the connection of rough sets with the 3-valued semantics being considered in this work. We end the section with a rough set semantics for  $\mathcal{L}_K$  (cf. Theorem 17), obtained as a consequence of the representation results of Section 4.1 below.

#### 4.1 Rough set representation of Kleene algebras

Algebraically, the collection  $\mathcal{D}$  of definable sets forms a complete atomic Boolean algebra in which atoms are the equivalence classes. The collection  $\mathcal{RS}$  forms a distributive lattice – in fact, it forms a Kleene algebra. On the other hand, observe that  $\mathcal{R}$  is the set  $\mathcal{D}^{[2]}$  and hence forms a Kleene algebra (cf. Proposition 1) as well.  $\mathcal{R}$  has earlier been studied, for instance, by Banerjee and Chakraborty in [22], and shown to form *topological quasi-Boolean*, *pre-rough* and *rough* algebras. Note that, for an approximation space  $(U, R)$ , as sets  $\mathcal{R}$  and  $\mathcal{RS}$  may not be the same. So, it is natural to ask how  $\mathcal{R}$  and  $\mathcal{RS}$  differ as algebraic structures. The following result mentioned in [22] gives a connection between the two. The proof is not given in [22]; we sketch it here, as it is used in the sequel.

**Theorem 13.** For any approximation space  $(U, R)$ , there exists an approximation space  $(U', R')$  such that  $\mathcal{R}$  corresponding to  $(U, R)$  is order isomorphic to  $\mathcal{R}'$  corresponding to  $(U', R')$ . Further,  $\mathcal{R}' = \mathcal{RS}'$ , the latter denoting the collection of rough sets in the approximation space  $(U', R')$ .

*Proof.* Let  $(U, R)$  be the given approximation space. Consider the set  $\mathbf{A} := \{a \in U : |R(a)| = 1\}$ , where  $R(a)$  denotes the equivalence class of  $a$  in  $U$ . So  $\mathbf{A}$  is the collection of all elements which are  $R$ -related only to themselves. Now, let us construct a set  $\mathbf{A}'$  which consists of ‘dummy’ elements not in  $U$ , indexed by the set  $\mathbf{A}$ , i.e.,  $\mathbf{A}' := \{a' : a \in \mathbf{A}\}$ . Let  $U' = U \cup \mathbf{A}'$ . Define an equivalence relation  $R'$  on  $U'$  as follows.



If  $a \in U$  then  $R'(a) := R(a) \cup \{x' \in \mathbf{A}' : x \in R(a) \cap \mathbf{A}\}$ .

If  $a' \in \mathbf{A}'$  then  $R'(a') := R(a) (= \{a, a'\})$ .

Note that the number of equivalence classes in both the approximation spaces is the same. Define the map  $\phi : \mathcal{R} \rightarrow \mathcal{R}'$  as  $\phi(D_1, D_2) := (D'_1, D'_2)$ , where  $D'_1 := D_1 \cup \{x' \in \mathbf{A}' : x \in D_1 \cap \mathbf{A}\}$  and  $D'_2 := D_2 \cup \{x' \in \mathbf{A}' : x \in D_2 \cap \mathbf{A}\}$ . Then  $\phi$  is an order isomorphism.  $\square$

Since  $\mathcal{R}$  and  $\mathcal{RS}$  for any approximation space  $(U, R)$  form Kleene algebras, Theorem 13 can easily be extended to Kleene algebras as follows.

**Theorem 14.** *Let  $(U, R)$  be an approximation space. There exists an approximation space  $(U', R')$  such that  $\mathcal{R}$  corresponding to  $(U, R)$  is Kleene isomorphic to  $\mathcal{RS}' (= \mathcal{R}')$  corresponding to  $(U', R')$ .*

*Proof.* Consider  $(U', R')$  and  $\phi$  as in Theorem 13.  $\phi$  is a lattice isomorphism, as the restriction of  $\phi$  to the completely join irreducible elements of the lattices  $\mathcal{D}^{[2]}$  and  $\mathcal{D}'^{[2]}$  is an order isomorphism (using Proposition 3 and Lemma 1). Let us now show that  $\phi(\sim(D_1, D_2)) = \sim(\phi(D_1, D_2))$ . To avoid confusion, we follow these notations: for  $X \subseteq U$  we use  $X^{c_1}$  for the complement in  $U$  and  $X^{c_2}$  for the complement in  $U'$ .

Now,  $\phi(\sim(D_1, D_2)) = \phi(D_2^{c_1}, D_1^{c_1}) = ((D_2^{c_1})', (D_1^{c_1})')$ . By definition of  $\phi$ , we have:

$$\begin{aligned} (D_2^{c_1})' &= D_2^{c_1} \cup \{x' \in \mathbf{A}' : x \in D_2^{c_1} \cap \mathbf{A}\}. \\ (D_1^{c_1})' &= D_1^{c_1} \cup \{x' \in \mathbf{A}' : x \in D_1^{c_1} \cap \mathbf{A}\}. \end{aligned}$$

*Claim.*  $(D_2^{c_1})' = (D'_2)^{c_2}$ , and  $(D_1^{c_1})' = (D'_1)^{c_2}$ .

*Proof of Claim:* Let us first prove that

$$\begin{aligned} (D_2^{c_1})' &= D_2^{c_1} \cup \{x' \in \mathbf{A}' : x \in D_2^{c_1} \cap \mathbf{A}\} = (D'_2)^{c_2} = (D_2 \cup \{x' : x \in D_2 \cap \mathbf{A}\})^{c_2} = \\ &= (D_2)^{c_2} \cap (\{x' \in \mathbf{A}' : x \in D_2 \cap \mathbf{A}\})^{c_2}. \end{aligned}$$

Let  $X := \{x' \in \mathbf{A}' : x \in D_2^{c_1} \cap \mathbf{A}\}$  and  $Y := \{x' \in \mathbf{A}' : x \in D_2 \cap \mathbf{A}\}$ .

Consider  $a \in (D_2^{c_1})' = D_2^{c_1} \cup X$ .

Case 1  $a \in D_2^{c_1}$ :

As,  $D_2 \subseteq U$ ,  $D_2^{c_1} \subseteq D_2^{c_2}$ . Hence  $a \in D_2^{c_2}$ . As  $D_2^{c_1} \subseteq U$ ,  $a \notin \mathbf{A}'$ , whence  $a \in Y^{c_2}$ . So  $a \in (D'_2)^{c_2}$ .

Case 2  $a \in X$ :

$a = x'$ , where  $x \in D_2^{c_1} \cap \mathbf{A}$ . As,  $x' \in R'(x)$  and  $D_2^{c_2}$  is the union of equivalence classes, in particular it contains  $R'(x)$ . So  $a = x' \in D_2^{c_2}$ .

$x \in D_2^{c_1}$  implies  $x \notin D_2$ . Hence  $a = x' \in Y^{c_2}$ . So  $a \in (D'_2)^{c_2}$ .

Conversely, let  $a \in (D'_2)^{c_2} = (D_2)^{c_2} \cap Y^{c_2}$ .

Case 1  $a \in U$ :

$a \in D_2^{c_2} \Rightarrow a \in D_2^{c_1}$ . Hence  $a \in (D_2^{c_1})'$ .

Case 2  $a \in \mathbf{A}'$ :

$a \in Y^{c_2}$  implies  $a \in \{x' \in \mathbf{A}' : x \in D_2^{c_1} \cap \mathbf{A}\}$ . Hence  $a \in (D_2^{c_1})'$ .

Similar arguments as above show that  $(D_1^{c_1})' = (D'_1)^{c_2}$ .

*Proof of Theorem 14:*

$$\phi(\sim(D_1, D_2)) = \phi(D_2^{c_1}, D_1^{c_1}) = ((D_2^{c_1})', (D_1^{c_1})') = ((D'_2)^{c_2}, (D'_1)^{c_2}) = \sim\phi(D_1, D_2).$$

Hence  $\phi$  is a Kleene isomorphism.  $\square$

It is now not hard to see the correspondence between a complete atomic Boolean algebra and rough sets in an approximation space.

**Theorem 15.** *Let  $B$  be a complete atomic Boolean algebra.*

- (i) *There exists an approximation space  $(U, R)$  such that*
  - (a)  *$B \cong \mathcal{D}$ .*
  - (b)  *$B^{[2]}$  is Kleene isomorphic to  $\mathcal{R}$ .*
- (ii) *There exists an approximation space  $(U', R')$  such that  $B^{[2]}$  is Kleene isomorphic to  $\mathcal{RS}'$ .*

*Proof.* Let  $U$  denote the collection of all atoms of  $B$ , and  $R$  the identity relation on  $U$ .  $(U, R)$  is the required approximation space.  $\square$

Thus Theorem 2 can be rephrased in terms of rough sets, and we get Theorem 1(ii).

**Corollary 2.** *Given a Kleene algebra  $\mathcal{K}$ , there exists an approximation space  $(U, R)$  such that  $\mathcal{K}$  can be embedded into  $\mathcal{RS}$ . In other words, every Kleene algebra is isomorphic to an algebra of rough sets in a Pawlak approximation space.*

## 4.2 Rough sets and the Kleene algebra **3**

The definitions  $(*)$  of lower and upper approximations of a set  $A$  in an approximation space  $(U, R)$  (cf. beginning of Section 4), immediately yield the following interpretations.

1.  $x$  *certainly* belongs to  $A$ , if  $x \in \text{LA}$ , i.e. all objects which are indiscernible to  $x$  are in  $A$ .
2.  $x$  *certainly does not* belong to  $A$ , if  $x \notin \text{UA}$ , i.e. all objects which are indiscernible to  $x$  are not in  $A$ .
3. Belongingness of  $x$  to  $A$  is *not certain, but possible*, if  $x \in \text{UA}$  but  $x \notin \text{LA}$ . In rough set terminology, this is the case when  $x$  is in the *boundary* of  $A$ : some objects indiscernible to  $x$  are in  $A$ , while some others, also indiscernible to  $x$ , are in  $A^c$ .

These interpretations have led to much work in the study of connections between 3-valued algebras or logics and rough sets, see for instance [23–28]. In particular, in [26], Avron and Konikowska have obtained a non-deterministic logical matrix and studied the 3-valued logic generated by this matrix, which is a sort of First order logic. In the direction of propositional logic, Banerjee and Chakraborty in [22, 23] obtained *pre-rough logic* for the class of pre-rough algebras. It was subsequently proved by Banerjee in [23] that 3-valued Łukasiewicz logic and pre-rough logic are equivalent, thereby imparting a rough set semantics to the former.

Let us spell out the natural connections of the Kleene algebra **3** with rough sets.

Observe that **3**, being isomorphic to **2**<sup>[2]</sup> (as noted earlier), can also be viewed as a collection of rough sets in an approximation space, due to Theorem 15(ii).

**Proposition 4.** *There exists an approximation space  $(U, R)$  such that  $\mathbf{3} \cong \mathcal{RS}$ .*

*Proof.* Let  $U := \{x, y\}$  and consider the equivalence relation  $R := U \times U$  on  $U$ . The correspondence is depicted in Figure 7.

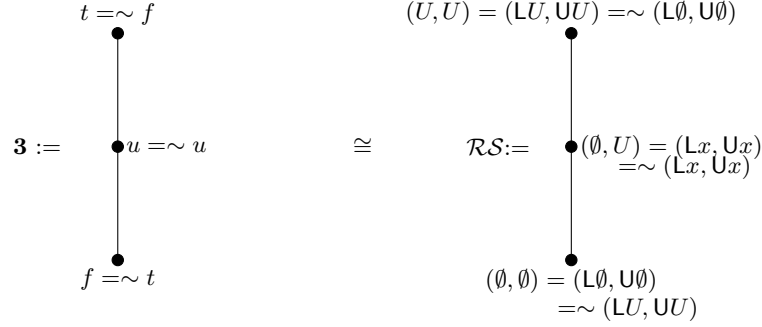


Fig. 7.  $\mathbf{3} \cong \mathcal{RS}$

□

On the other hand, interpretations 1-3 above give rise to a correspondence with the set  $\mathbf{3} := \{f, u, t\}$ , that assigns to every  $x \in U$  and rough set  $(\mathbf{L}A, \mathbf{U}A)$  in  $(U, R)$ , the value  $t$  when  $x \in \mathbf{L}A$ ,  $u$  when  $x \in \mathbf{U}A \setminus \mathbf{L}A$ , and  $f$  in case  $x \notin \mathbf{U}A$ . As one can see, this is akin to the valuation defined in the proof of Theorem 11. In fact, using results of the previous sections, we can formally link the 3-valued semantics being considered here, and rough sets.

Let  $\mathcal{A}_{K\mathcal{RS}}$  denote the class containing the collections  $\mathcal{RS}$  of rough sets over all possible approximation spaces  $(U, R)$ .

**Theorem 16.** *For any  $\alpha, \beta \in \mathcal{F}$ ,*

- (i)  $\alpha \models_{\mathcal{A}_K} \beta$  if and only if  $\alpha \models_{\mathcal{A}_{K\mathcal{RS}}} \beta$ ,
- (ii)  $\alpha \models_{\mathcal{A}_{K\mathcal{RS}}} \beta$  if and only if  $\alpha \models_{t,f} \beta$ .

In the process, we have obtained a rough set semantics for  $\mathcal{L}_K$ .

**Theorem 17.** *For any  $\alpha, \beta \in \mathcal{F}$ ,  $\alpha \vdash_{\mathcal{L}_K} \beta$  if and only if  $\alpha \models_{\mathcal{A}_{K\mathcal{RS}}} \beta$ .*

## 5 Perp semantics for the logic $\mathcal{L}_K$

Dunn's framework of perp semantics for negations is of logical, philosophical as well as algebraic importance. On the one hand, it provides relational semantics for various logics with negations (cf. e.g., [6, 14]), interpreting the negations as 'impossibility' or 'unnecessity' operators. On the other hand, one can give representations of various algebras as set algebras [29]. In this section, we characterize

the Kleene consequent  $\alpha \wedge \sim \alpha \vdash \beta \vee \sim \beta$  in Dunn's framework of negations. Further, one obtains a representation of Kleene algebras through duality. First, we briefly present the basics of perp semantics. For details, one may refer to [29, 20, 6, 14].

**Definition 8.** A compatibility frame is a triple  $(W, C, \leq)$  with the following properties.

1.  $(W, \leq)$  is a partially ordered set.
2.  $C$  is a binary relation on  $W$  such that for  $x, y, x', y' \in W$ , if  $x' \leq x$ ,  $y' \leq y$  and  $xCy$  then  $x'Cy'$ .

$C$  is called a compatibility relation on  $W$ .

A perp frame is a tuple  $(W, \perp, \leq)$ , where  $\perp$ , the perp relation on  $W$ , is the complement of the compatibility relation  $C$ .

Recall the syntax of the logic  $\mathcal{L}_K$  as defined in Section 3.

**Definition 9.** A relation  $\models$  between points of  $W$  and propositional variables in  $\mathcal{P}$  is called an evaluation, if it satisfies the hereditary condition:

- if  $x \models p$  and  $x \leq y$  then  $y \models p$ .

Recursively, an evaluation can be extended to  $\mathcal{F}$  as follows.

1.  $x \models \alpha \wedge \beta$  iff  $x \models \alpha$  and  $x \models \beta$ .
2.  $x \models \alpha \vee \beta$  iff  $x \models \alpha$  or  $x \models \beta$ .
3.  $x \models \top$ .
4.  $x \not\models \perp$ .
5.  $x \models \sim \alpha$  iff  $\forall y(xCy \Rightarrow y \not\models \alpha)$ .

Then one can easily show that  $\models$  satisfies the hereditary condition for all formulae in  $\mathcal{F}$ . For the compatibility frame  $\mathbf{F} := (W, C, \leq)$ , the pair  $(\mathbf{F}, \models)$  for an evaluation  $\models$  is called a *model*. The notion of validity is introduced next in the following (usual) manner.

A consequent  $\alpha \vdash \beta$  is *valid in a model*  $(\mathbf{F}, \models)$ , denoted as  $\alpha \models_{(\mathbf{F}, \models)} \beta$ , if and only if, if  $x \models \alpha$  then  $x \models \beta$ , for each  $x \in W$ .

$\alpha \vdash \beta$  is *valid in the compatibility frame*  $\mathbf{F}$ , denoted as  $\alpha \models_{\mathbf{F}} \beta$ , if and only if  $\alpha \models_{(\mathbf{F}, \models)} \beta$  for every model  $(\mathbf{F}, \models)$ .

Let  $\mathbb{F}$  denote a class of compatibility frames.  $\alpha \vdash \beta$  is *valid in*  $\mathbb{F}$ , denoted as  $\alpha \models_{\mathbb{F}} \beta$ , if and only if  $\alpha \models_{\mathbf{F}} \beta$  for every frame belonging to  $\mathbb{F}$ .

Following [14], let  $K_i$  denote the logic whose postulates are 1 to 12 of the logic  $\mathcal{L}_K$  (cf. Definition 4). In [14] it has been proved that  $K_i$  is the minimal logic which is sound and complete with respect to the class of all compatibility frames. Frame completeness results for various normal logics with negation have been proved using the canonical frames for the logics. Let us we give the definitions for the canonical frame.

**Definition 10.** A set of sentences  $P$  in a logic  $\Lambda$  is called a prime theory if

1.  $\alpha \vdash \beta$  holds and  $\alpha \in P$ , then  $\beta \in P$ .
2.  $\alpha, \beta \in P$  then  $\alpha \wedge \beta \in P$ .
3.  $\top \in P$  and  $\perp \notin P$ .
4.  $\alpha \vee \beta \in P$  implies  $\alpha \in P$  or  $\beta \in P$ .

Let  $W_c$  be the collection of all non-trivial prime theories of  $\Lambda$ . Define a relation  $C_c$  on  $W_c$  as  $P_1 C_c P_2$  if and only if, for all sentences  $\alpha$  of  $\Lambda$ ,  $\sim \alpha \in P_1$  implies  $\alpha \notin P_2$ . The tuple  $(W_c, C_c, \subseteq)$  is the *canonical frame* for  $\Lambda$ .

A logic  $\Lambda$  is called *canonical*, if its canonical frame is a frame for  $\Lambda$ .

Let us now consider the canonical frame for the logic  $\mathcal{L}_K$ . Note that  $\mathcal{L}_K$  contains  $K_i$  along with the postulates (13)  $\alpha \vdash \sim \sim \alpha$ , (14)  $\sim \sim \alpha \vdash \alpha$  and (15)  $\alpha \wedge \sim \alpha \vdash \beta \vee \sim \beta$  of Definition 4. The consequents  $\alpha \vdash \sim \sim \alpha$  and  $\sim \sim \alpha \vdash \alpha$  have been characterized by Dunn (for e.g., [14]) and Restall [30] respectively as follows.

**Theorem 18.**

1.  $\alpha \vdash \sim \sim \alpha$  is valid in the class of all compatibility frames satisfying the following frame condition:

$$\forall x \forall y (xCy \rightarrow yCx).$$

2.  $\sim \sim \alpha \vdash \alpha$  is valid in the class of all compatibility frames satisfying the frame condition:

$$\forall x \exists y (xCy \wedge \forall z (yCz \rightarrow z \leq x)).$$

3. The canonical frame for  $\mathcal{L}_K$  satisfies both the above frame conditions.

It remains for us to characterize the Kleene consequent  $\alpha \wedge \sim \alpha \vdash \beta \vee \sim \beta$  with an appropriate frame condition, and prove that the canonical frame for  $\mathcal{L}_K$  satisfies the condition.

**Theorem 19.**  $\alpha \wedge \sim \alpha \vdash \beta \vee \sim \beta$  is valid in a compatibility frame, if and only if the compatibility relation satisfies the following first order property:

$$\forall x (xCx \vee \forall y (xCy \rightarrow y \leq x)). \quad (*)$$

The canonical frame for  $\mathcal{L}_K$  satisfies (\*).

*Proof.* Consider any compatibility frame  $(W, C, \leq)$ , let (\*) hold, and let  $x \in W$ . Suppose  $xCx$ , then  $x \not\models \alpha \wedge \sim \alpha$ , and trivially, if  $x \models \alpha \wedge \sim \alpha$  then  $x \models \beta \vee \sim \beta$ .

Now suppose  $\forall y (xCy \rightarrow y \leq x)$  is true. Let  $x \not\models \beta$  and  $xCz$ . Then  $z \leq x$ , whence  $z \not\models \beta$ . So, by definition  $x \models \sim \beta$ . Hence  $x \models \beta \vee \sim \beta$ . Hence in any case if  $x \models \alpha \wedge \sim \alpha$  then  $x \models \beta \vee \sim \beta$ .

Let (\*) not hold.

This implies  $\exists x (\text{not}(xCx) \wedge \exists y (xCy \wedge y \not\leq x))$ . Let us define, for any  $z, w \in W$ ,

$$z \models p \text{ if and only if } x \leq z \text{ and } \sim xCz,$$

$$w \models q \text{ if and only if } y \leq w.$$

We show that  $\models$  is an evaluation. Let  $z \models p$  and  $z \leq z'$ . Then  $x \leq z'$ . If  $xCz'$ , then by the frame condition on  $C$  we have

$$x \leq x, z \leq z' \text{ } xCz' \text{ imply } xCz,$$

which is a contradiction to the fact that  $z \models p$ .

Furthermore,  $x \models p$ , as  $x \leq x$  and  $\text{not}(xCx)$ . We also have  $x \models \sim p$ : if  $xCw$  for any  $w \in W$  then by definition,  $w \not\models p$ . Hence,  $x \models \sim p$  and so  $x \models p \wedge \sim p$ . On the other hand,  $x \not\models q$  as  $y \not\leq x$ . By the assumption,  $xCy$  and  $y \models q$ , hence  $x \not\models \sim q$ . So, we have  $x \models p \wedge \sim p$  but  $x \not\models q \vee \sim q$ .

*Canonicity :*

Let  $\sim PC_cP$ . By definition of  $C_c$ , then there exists an  $\alpha \in \mathcal{F}$  such that  $\alpha, \sim \alpha \in P$ , but then  $\alpha \wedge \sim \alpha \in P$ . Hence for all  $\beta \in \mathcal{F}$ ,  $\beta \vee \sim \beta \in P$ . So, for all  $\beta$ , either  $\beta \in P$  or  $\sim \beta \in P$ .

Now let  $PC_cQ$  and let  $\beta \in Q$ . Hence  $\sim \beta \notin P$  but as from above,  $\sim PC_cP$ , we have  $\beta \in P$ . So,  $Q \subseteq P$ .  $\square$

**Definition 11.** Let us call a compatibility frame  $(W, C, \leq)$  a Kleene frame if it satisfies the following frame conditions.

1.  $\forall x \forall y (xCy \rightarrow yCx)$ .
2.  $\forall x \exists y (xCy \wedge \forall z (yCz \rightarrow z \leq x))$ .
3.  $\forall x (xCx \vee \forall y (xCy \rightarrow y \leq x))$ .

Denote by  $\mathbb{F}_K$ , the class of all Kleene frames.

We have then arrived at

**Theorem 20.** The following are all equivalent, for any  $\alpha, \beta \in \mathcal{F}$ .

- (a)  $\alpha \vdash_{\mathcal{L}_K} \beta$ .
- (b)  $\alpha \models_{\mathcal{A}_K} \beta$ .
- (c)  $\alpha \models_{\mathcal{A}_{KRS}} \beta$ .
- (d)  $\alpha \models_{t,f} \beta$ .
- (e)  $\alpha \models_{\mathbb{F}_K} \beta$ .

## 6 Conclusions

In case of Boolean algebras and classical propositional logic, or De Morgan algebras and De Morgan logic, the algebraic semantics and the 2 or 4-valued semantics (respectively) are equivalent – due to representation theorems for the two classes of algebras. Here, analogously, we have the result for the class of Kleene algebras, that the algebraic semantics and a 3-valued semantics (given by  $\models_{t,f}$ ) of the logic  $\mathcal{L}_K$  of Kleene algebras are equivalent. This is due to the representation theorem (Theorem 2) one is able to prove: any Kleene algebra is embeddable in  $B^{[2]}$ , for some Boolean algebra  $B$ .

Moreover, we have proved that the 3-valued semantics of  $\mathcal{L}_K$  can translate into a rough set (Theorem 17) as well as perp semantics (Theorem 20). It is thus established that  $\mathcal{L}_K$  can be imparted equivalent semantics from different perspectives.

## References

1. Kalman, J.: Lattices with involution. *Transactions of American Mathematical Society* **87** (1958) 485–491
2. Cignoli, R.: Boolean elements in Łukasiewicz algebras i. *Proc. Japan Acad.* **41** (1965) 670–675
3. Cignoli, R.: The class of Kleene algebras satisfying an interpolation property and Nelson algebras. *Algebra Universalis* **23(3)** (1986) 262–292
4. Rasiowa, H.: *An Algebraic Approach to Non-classical Logics*. North-Holland (1974)
5. Dunn, J.: *The algebra of intensional logic*. Doctoral dissertation, University of Pittsburgh (1966)
6. Dunn, J.: A comparative study of various model-theoretic treatments of negation: A history of formal negations. In Gabbay, D., Wansing, H., eds.: *What is Negations?* Kluwer Academic Publishers, Netherlands (1999) 23–51
7. Cignoli, R.: The algebras of Łukasiewicz many-valued logic: A historical overview. In Aguzzoli, S., Ciabattoni, A., Gerla, B., Manara, C., Marra, V., eds.: *Algebraic and Proof-theoretic Aspects of Non-classical Logics*, LNAI 4460. Springer-Verlag, Berlin Heidelberg (2007) 69–83
8. Katriňák, T.: Construction of regular double p-algebras. *Bull. Soc. Roy. Sci. Liege* **43** (1974) 238–246
9. Boicescu, V., Filipoiu, A., Georgescu, G., Rudeanu, S.: *Łukasiewicz-Moisil Algebras*. North-Holland, Amsterdam (1991)
10. Pawlak, Z.: Rough sets. *International Journal of Computer and Information Sciences* **11** (1982) 341–356
11. Pawlak, Z.: *Rough Sets: Theoretical Aspects of Reasoning About Data*. Kluwer Academic Publishers (1991)
12. Comer, S.: Perfect extensions of regular double Stone algebras. *Algebra Universalis* **34(1)** (1995) 96–109
13. Järvinen, J., Radeleczki, S.: Representation of Nelson algebra by rough sets determined by quasiorder. *Algebra Universalis* **66** (2011) 163–179
14. Dunn, J.: Negation in the context of gaggle theory. *Studia Logica* **80** (2005) 235–264
15. Davey, B.A., Priestley, H.A.: *Introduction to Lattices and Order*. Cambridge University Press (2002)
16. Vakarelov, D.: Notes on N-lattices and constructive logic with strong negation. *Studia Logica* **36** (1977) 109–125
17. Fidel, M.: An algebraic study of prepositional system of Nelson. In Arruda, A.I., da Costa, N.C.A., Chuaqui, R., eds.: *Mathematical Logic: Proceedings of First Brazilian Conference*. Lecture Notes in Pure and Applied Mathematics Vol. 39, M.Dekker Inc., New York (1978) 99–117
18. Balbes, R., Dwinger, P.: *Distributive Lattices*. University of Missouri Press, Columbia (1974)
19. Birkhoff, G.: *Lattice Theory*. Colloquium Publications, Vol. XXV, 3rd edn., American Mathematical Society, Providence (1995)
20. Dunn, J.: Partiality and its dual. *Studia Logica* **66** (2000) 5–40
21. Urquhart, A.: Basic many-valued logic. In Gabby, D., Guenther, F., eds.: *Handbook of philosophical logic*, vol. 2. Springer Netherlands, Reidel, Dordrecht (2001) 249–295
22. Banerjee, M., Chakraborty, M.K.: Rough sets through algebraic logic. *Fundamenta Informaticae* **28(3-4)** (1996) 211–221

23. Banerjee, M.: Rough sets and 3-value Łukasiewicz logic. *Fundamenta Informaticae* **31** (1997) 213–220
24. Iturrioz, L.: Rough sets and three-valued structures. In Orłowska, ed.: *Logic at Work: Easy Dedicated to the Memory of Helena Rasiowa*, volume 24 of *Studies in Fuzziness and Soft Computing*. Springer Physica-Verlag (1999) 596–603
25. Pagliani, P.: Rough set theory and logic-algebraic structures. In Orłowska, ed.: *Incomplete Information: Rough Set analysis*, volume 3 of *Studies in Fuzziness and Soft Computing*. Springer Physica-Verlag (1998) 109–190
26. Avron, A., Konikowska, B.: Rough sets and 3-valued logics. *Studia Logica* **90** (2008) 69–92
27. Ciucci, D., Dubois, D.: Three-valued logics, uncertainty management and rough sets. In Peters, J., Skowron, A., eds.: *Transactions on Rough Sets XVII*. LNCS, vol. 8375. Springer-Heidelberg, Berlin Heidelberg (2014) 1–32
28. Duntsch, I.: A logic for rough sets. *Theoretical Computer Science* **179** (1997) 427–436
29. Dunn, J.: Star and perp: Two treatments of negation. In Tomberlin, J., ed.: *Philosophical Perspectives*, volume 7. Ridgeview Publishing Company, Atascadero, California (1994) 331–357
30. Restall, G.: Defining double negation elimination. *L. J. of IGPL* (**8**(6)) 853–860